

# Renormalization group approach to Sudakov resummation in prompt photon production

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## Abstract

We prove the all-order exponentiation of soft logarithmic corrections to prompt photon production in hadronic collisions, by generalizing an approach previously developed in the context of Drell-Yan production and deep-inelastic scattering. We show that all large logs in the soft limit can be expressed in terms of two dimensionful variables, and we use the renormalization group to resum them. The resummed results that we obtain are more general though less predictive than those proposed by other groups, in that they can accommodate for violations of Sudakov factorization.

# 1 Introduction

The resummation of logarithmically-enhanced terms in the perturbative expansion of QCD cross sections near the boundary of phase space (Sudakov [1] resummation, henceforth) has been proved to next-to-leading logarithmic order for deep-inelastic scattering in the  $x \rightarrow 1$  limit and for Drell-Yan production near threshold [2, 3]. In fact, resummation formulae can be derived for a wider class of processes and to all logarithmic orders, by assuming the validity of a suitable two-scale factorization [4]. The two scales correspond to the hard scale which guarantees the perturbative nature of the process, and another dimensionful variable, whose large ratio to the first perturbative scale is the quantity whose log must be resummed to all orders. This two-scale factorization, originally derived for semi-inclusive two-jet production in  $e^+e^-$  annihilation [5], can be proved for deep-inelastic scattering and Drell-Yan in the  $\phi^3$  theory in six space-time dimensions [6], but its generalization to QCD processes is nontrivial.

In ref. [7] a new approach to Sudakov resummation was developed, and applied to the Drell-Yan and deep-inelastic scattering processes. This approach has the advantage of being valid to all logarithmic orders, and self-contained, in that it does not require any factorization beyond the standard factorization of collinear singularities. It relies on an essentially kinematical analysis of the phase space for the given process in the soft limit, which is used to establish the result that the dependence on the resummation variable only appears through a given fixed dimensionful combination. This provides a second dimensionful variable, along with the hard scale of the process, which can be resummed using standard renormalization group techniques. Beyond the leading log level, the resummed result found within this approach turns out to be somewhat less predictive than the result of refs. [2, 6]: in those references the next<sup>k</sup>-to-leading log resummed result is fully determined by a fixed next<sup>k</sup>-to-leading order computation, whereas a higher fixed order computation is needed to determine all coefficients in the resummed formula of ref. [7]. The more predictive result is recovered within this approach if the dependence of the perturbative coefficients on the two dimensionful variables factorizes, i.e. if the two-scale factorization mentioned above holds.

In this paper we discuss the generalization of the approach of ref. [7] to the resummation of the inclusive transverse momentum spectrum of prompt photons produced in hadronic collisions in the region where the transverse momentum is close to its maximal value. Prompt photon production is a less inclusive process than Drell-Yan or deep-inelastic scattering, and it is especially interesting from the point of view of the approach of ref. [7], because the large logs which must be resummed turn out to depend on *two* independent dimensionful variables, on top of the hard scale of the process: hence, prompt photon production is characterized by three scales. The possibility that the general factorization ref. [6] might extend to prompt photon production was discussed in ref. [8], based on previous generalizations [9] of factorization, and used to derive the corresponding resummed results. Resummation formulae for prompt photon production in the approach of ref. [2] were also proposed in ref. [10], and some arguments which might support such resummation were presented in ref. [11]. Our treatment will provide a full proof of resummation to all logarithmic orders. Our resummation formula does not require the factorization proposed in refs. [8, 10], and it is accordingly less predictive. Because of the presence of two scales, it is also weaker than the result of ref. [7] for DIS and Drell-Yan production. Increasingly more

predictive results are recovered if increasingly restrictive forms of factorization hold.

## 2 Kinematics of prompt photon production in the soft limit

We consider the process

$$H_1(P_1) + H_2(P_2) \rightarrow \gamma(p_\gamma) + X, \quad (2.1)$$

specifically the differential cross section  $p_\perp^3 \frac{d\sigma}{dp_\perp}$ , where  $p_\perp$  is the transverse momentum of the photon with respect to the direction of the colliding hadrons  $H_1$  and  $H_2$ , and

$$x = \frac{4p_\perp^2}{S}; \quad S = (P_1 + P_2)^2; \quad 0 \leq x_\perp \leq 1. \quad (2.2)$$

The factorized expression for this cross section in perturbative QCD is

$$p_\perp^3 \frac{d\sigma}{dp_\perp}(x, p_\perp^2) = \sum_{a,b} \int_0^1 dx_1 dx_2 dz x_1 F_a^{H_1}(x_1, \mu^2) x_2 F_b^{H_2}(x_2, \mu^2) C_{ab} \left( z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \delta(x - zx_1 x_2), \quad (2.3)$$

where  $F_a^{H_1}(x_1, \mu^2)$ ,  $F_b^{H_2}(x_2, \mu^2)$  are the distribution functions of partons  $a, b$  in the colliding hadrons, we have defined

$$Q^2 = 4p_\perp^2, \quad (2.4)$$

$$z = \frac{Q^2}{s}, \quad 0 \leq z \leq 1, \quad (2.5)$$

where  $s$  is the center-of-mass energy of the partonic process, and the coefficient function  $C_{ab}(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2))$  is defined in terms of the partonic cross section for the process where partons  $a, b$  are incoming as

$$C_{ab} \left( z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) = p_\perp^3 \frac{d\hat{\sigma}_{ab}}{dp_\perp}. \quad (2.6)$$

We will study the cross section eq. (2.3) in the limit in which  $z \rightarrow 1$ , i.e., the transverse momentum of the photon is close to its maximal value, thereby suppressing the phase space for further parton radiation. The convolution in eq. (2.3) is turned into an ordinary product by Mellin transformation:

$$\sigma(N, Q^2) = \sum_{a,b} F_a^{H_1}(N+1, \mu^2) F_b^{H_2}(N+1, \mu^2) C_{ab}(N, Q^2/\mu^2, \alpha_s(\mu^2)), \quad (2.7)$$

where

$$\sigma(N, Q^2) = \int_0^1 dx x^{N-1} p_\perp^3 \frac{d\sigma}{dp_\perp}(x, p_\perp^2) \quad (2.8)$$

and similarly for  $C_{ab}$  and  $F_i^{H_j}$ .

Whereas the cross section  $\sigma(N, Q^2)$  is clearly  $\mu^2$ -independent, this is not the case for each contribution to it from individual parton subprocesses. However, the  $\mu^2$  dependence of each

contribution to the sum over  $a, b$  in eq. (2.3) is proportional to the off-diagonal anomalous dimensions  $\gamma_{qg}$  and  $\gamma_{gq}$ . In the large  $N$  limit, these are suppressed by a power of  $\frac{1}{N}$  in comparison to  $\gamma_{gg}$  and  $\gamma_{qq}$ , or, equivalently, the corresponding splitting functions are suppressed by a factor of  $1 - x$  in the large  $x$  limit. Hence, in the large  $N$  limit each parton subprocess can be treated independently, specifically, each  $C_{ab}$  is separately renormalization-group invariant. Because we are interested in the behaviour of  $C_{ab}(N, Q^2/\mu^2, \alpha_s(\mu^2))$  in the limit  $N \rightarrow \infty$ , we shall henceforth omit the parton indices  $a, b$  and assume that each subprocess is being treated independently. Furthermore, on top of eqs. (2.3, 2.7) the physical process eq. (2.1) receives another factorized contribution, in which the final-state photon is produced by fragmentation of a primary parton produced in the partonic sub-process. However, the cross section for this process is also suppressed by a factor of  $\frac{1}{N}$  in the large  $N$  limit. This is due to the fact that the fragmentation function carries this suppression, for the same reason why the anomalous dimensions  $\gamma_{qg}$  and  $\gamma_{gq}$  are suppressed. Therefore, we will disregard the fragmentation contribution.

Because resummation takes the form of an exponentiation, it is useful to introduce the log derivative of the cross section  $\sigma$ , *i.e.*, the so-called physical anomalous dimension defined as

$$Q^2 \frac{\partial \sigma(N, Q^2)}{\partial Q^2} = \gamma(N, \alpha_s(Q^2)) \sigma(N, Q^2). \quad (2.9)$$

The physical anomalous dimension  $\gamma$  eq. (2.9) is independent of factorization scale, and it is related to the standard anomalous dimension  $\gamma^{AP}$ , defined by

$$\mu^2 \frac{\partial F(N, \mu^2)}{\partial \mu^2} = \gamma^{AP}(N, \alpha_s(\mu^2)) F(N, \mu^2), \quad (2.10)$$

according to

$$\gamma(N, \alpha_s(Q^2)) = \frac{\partial \ln C(N, Q^2/\mu^2, \alpha_s(\mu^2))}{\partial \ln Q^2} = \gamma^{AP}(N, \alpha_s(Q^2)) + \frac{\partial \ln C(N, 1, \alpha_s(Q^2))}{\partial \ln Q^2}, \quad (2.11)$$

where we have schematically denoted by  $F(N, \mu^2)$  the product  $F_a^{H_1}(N, \mu^2) F_b^{H_2}(N, \mu^2)$  that appears in the cross section for the given parton subprocess, eq. (2.7). In terms of the physical anomalous dimension, the cross section can be written as

$$\sigma(N, Q^2) = K(N; Q_0^2, Q^2) \sigma(N, Q_0^2) = \exp \left[ E(N; Q_0^2, Q^2) \right] \sigma(N, Q_0^2), \quad (2.12)$$

where

$$\begin{aligned} E(N; Q_0^2, Q^2) &= \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma(N, \alpha_s(k^2)) \\ &= \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{AP}(N, \alpha_s(k^2)) + \ln C(N, 1, \alpha_s(Q^2)) - \ln C(N, 1, \alpha_s(Q_0^2)). \end{aligned} \quad (2.13)$$

In the large- $x$  limit, the order- $n$  coefficient of the perturbative expansion is dominated by terms proportional to  $\left[ \frac{\ln^k(1-x)}{1-x} \right]_+$ , with  $k \leq 2n - 1$ , that must be resummed to all orders. Upon

Mellin transformation, these lead to contributions proportional to powers of  $\ln \frac{1}{N}$ . In the sequel, we will consider the resummation of these contributions to all logarithmic orders, and disregard all contributions to the cross section which are suppressed by powers of  $1 - x$ , i.e., upon Mellin transformation, by powers of  $\frac{1}{N}$ .

The resummation is performed in two steps, in analogy to ref. [7]. First, we show that the origin of the large logs is essentially kinematical: we identify the configurations which contribute in the soft limit, we show by explicit computation that large Sudakov logs are produced by the phase-space for real emission with the required kinematics as logs of two dimensionful variables, and we show that this conclusion is unaffected by virtual corrections. Second, we resum the logs of these variables using the renormalization group.

The  $l$ -th order correction to the leading  $O(\alpha_s)$  partonic process receives contribution from the emission of up to  $l + 1$  massless partons with momenta  $k_1, \dots, k_{l+1}$ , and

$$p_1 + p_2 = p_\gamma + k_1 + \dots k_{l+1}. \quad (2.14)$$

In the partonic center-of-mass frame, where

$$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1) \quad p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1) \quad (2.15)$$

$$p_\gamma = (p_\perp \cosh \eta_\gamma, \vec{p}_\perp, p_\perp \sinh \eta_\gamma), \quad (2.16)$$

we have

$$(p_1 + p_2 - p_\gamma)^2 = \frac{Q^2}{x}(1 - \sqrt{x} \cosh \eta_\gamma) = \sum_{i,j=1}^{l+1} k_i^0 k_j^0 (1 - \cos \theta_{ij}) \geq 0, \quad (2.17)$$

where  $\theta_{ij}$  is the angle between  $\vec{k}_i$  and  $\vec{k}_j$ . Hence,

$$1 \leq \cosh \eta_\gamma \leq \frac{1}{\sqrt{x}}. \quad (2.18)$$

Therefore,

$$\sum_{i,j=1}^{l+1} k_i^0 k_j^0 (1 - \cos \theta_{ij}) = \frac{Q^2}{2}(1 - x) + O[(1 - x)^2]. \quad (2.19)$$

Equation (2.19) implies that in the soft limit the sum of scalar products of momenta  $k_i$  of emitted partons eq. (2.17) must vanish. However, contrary to the case of deep-inelastic scattering or Drell-Yan, not all momenta  $k_i$  of the emitted partons can be soft as  $x \rightarrow 1$ , because the 3-momentum of the photon must be balanced. Assume thus that momenta  $k_i, i = 1, \dots, n; n < l+1$  are soft in the  $x \rightarrow 1$  limit, while momenta  $k_i, i > n$  are non-soft. For the sake of simplicity, we relabel non-soft momenta as

$$k'_j = k_{n+j}; \quad 1 \leq j \leq m+1; \quad m = l - n. \quad (2.20)$$

The generic kinematic configuration in the  $x = 1$  limit is then

$$\begin{aligned} k_i &= 0 & 1 \leq i \leq n \\ \theta_{ij} &= 0; & \sum_{j=1}^{m+1} k_j'^0 = p_\perp & 1 \leq i, j \leq m+1. \end{aligned} \quad (2.21)$$

for all  $n$  between 1 and  $l$ , namely, the configuration where at least one momentum is not soft, and the remaining momenta are either collinear to it, or soft.

With this labeling of the momenta, the phase space can be written as (see the Appendix of ref. [7])

$$\begin{aligned} & d\phi_{n+m+2}(p_1 + p_2; p_\gamma, k_1, \dots, k_n, k'_1, \dots, k'_{m+1}) \\ &= \int_0^s \frac{dq^2}{2\pi} d\phi_{n+1}(p_1 + p_2; q, k_1, \dots, k_n) \int_0^{q^2} \frac{dk'^2}{2\pi} d\phi_{m+1}(k'; k'_1, \dots, k'_{m+1}) d\phi_2(q; p_\gamma, k'). \end{aligned} \quad (2.22)$$

We shall now compute the phase space in the  $x \rightarrow 1$  limit. Consider first the two-body phase space  $d\phi_2$  in eq. (2.22). In the rest frame of  $q$  we have

$$\begin{aligned} d\phi_2(q; p_\gamma, k') &= \frac{d^{d-1}k'}{(2\pi)^{d-1}2k'^0} \frac{d^{d-1}p_\gamma}{(2\pi)^{d-1}2p_\gamma^0} (2\pi)^d \delta^{(d)}(q - k' - p_\gamma) \\ &= \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{P^{1-2\epsilon}}{\sqrt{q^2}} \sin^{-2\epsilon} \theta_\gamma d|\vec{p}_\gamma| d\cos\theta_\gamma \delta(|\vec{p}_\gamma| - P), \end{aligned} \quad (2.23)$$

where  $d = 4 - 2\epsilon$  and

$$P = \frac{\sqrt{q^2}}{2} \left( 1 - \frac{k'^2}{q^2} \right). \quad (2.24)$$

Because momenta  $k_i$ ,  $i \leq n$  are soft, up to terms suppressed by powers of  $1 - x$ , the rest frame of  $q$  is the same as the center-of-mass frame of the incoming partons, in which

$$|\vec{p}_\gamma| = p_\perp \cosh \eta_\gamma \quad (2.25)$$

$$\cos \theta_\gamma = \tanh \eta_\gamma. \quad (2.26)$$

Hence,

$$d\phi_2(q; p_\gamma, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{(Q^2/4)^{-\epsilon}}{\sqrt{q^2}} dp_\perp d\eta_\gamma \delta \left( \cosh \eta_\gamma - \frac{2P}{\sqrt{Q^2}} \right). \quad (2.27)$$

The conditions

$$\cosh \eta_\gamma = \frac{2P}{\sqrt{Q^2}} \geq 1; \quad k'^2 \geq 0 \quad (2.28)$$

restrict the integration range to

$$Q^2 \leq q^2 \leq s \quad (2.29)$$

$$0 \leq k'^2 \leq q^2 - \sqrt{Q^2 q^2}. \quad (2.30)$$

It is now convenient to define new variables  $u, v$

$$q^2 = Q^2 + u(s - Q^2) = Q^2 [1 + u(1 - x)] + O((1 - x)^2) \quad (2.31)$$

$$k'^2 = v(q^2 - \sqrt{Q^2 q^2}) = Q^2 \frac{1}{2} uv(1 - x) + O((1 - x)^2) \quad (2.32)$$

$$0 \leq u \leq 1 \quad ; \quad 0 \leq v \leq 1, \quad (2.33)$$

in terms of which

$$P = \frac{\sqrt{Q^2}}{2} \left[ 1 + \frac{1}{2}u(1-v)(1-x) \right] + O[(1-x)^2]. \quad (2.34)$$

Thus, the two-body phase space eq. (2.27) up to subleading terms is given by

$$d\phi_2(q; p_\gamma, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{(Q^2/4)^{-\epsilon}}{\sqrt{Q^2}} dp_\perp d\eta_\gamma \frac{\delta(\eta_\gamma - \eta_+) + \delta(\eta_\gamma - \eta_-)}{\sqrt{u(1-v)(1-x)}}, \quad (2.35)$$

where

$$\eta_\pm = \ln \left( \frac{2P}{\sqrt{Q^2}} \pm \sqrt{\frac{4P^2}{Q^2} - 1} \right) = \pm \sqrt{u(1-v)(1-x)}. \quad (2.36)$$

We now note that the phase-space element  $d\phi_{n+1}(p_1 + p_2; q, k_1, \dots, k_n)$  contains in the final state a system with large invariant mass  $q^2 \geq Q^2$ , plus a collection of  $n$  soft partons: this same configuration is encountered in the case of Drell-Yan pair production in the limit  $x_{DY} = q^2/s \rightarrow 1$ , discussed in ref. [7]. Likewise, the phase space for the set of collinear partons  $d\phi_{m+1}(k'; k'_1, \dots, k'_{m+1})$  is the same as the phase space for deep-inelastic scattering, where the invariant mass of the initial state  $k'^2$  vanishes as  $1-x$  (see eq. (2.32)). We may therefore use the results obtained in ref. [7]:<sup>1</sup>

$$d\phi_{n+1}(p_1 + p_2; q, k_1, \dots, k_n) = 2\pi \left[ \frac{N(\epsilon)}{2\pi} \right]^n (q^2)^{-n(1-\epsilon)} (s - q^2)^{2n-1-2n\epsilon} d\Omega^{(n)}(\epsilon) \quad (2.37)$$

$$d\phi_{m+1}(k'; k'_1, \dots, k'_{m+1}) = 2\pi \left[ \frac{N(\epsilon)}{2\pi} \right]^m (k'^2)^{m-1-m\epsilon} d\Omega'^{(m)}(\epsilon), \quad (2.38)$$

where  $N(\epsilon) = 1/(2(4\pi)^{2-2\epsilon})$  and

$$d\Omega^{(n)}(\epsilon) = d\Omega_1 \dots d\Omega_n \int_0^1 dz_n z_n^{(n-2)+(n-1)(1-2\epsilon)} (1-z_n)^{1-2\epsilon} \dots \int_0^1 dz_2 z_2^{1-2\epsilon} (1-z_2)^{1-2\epsilon} \quad (2.39)$$

$$d\Omega'^{(m)}(\epsilon) = d\Omega'_1 \dots d\Omega'_m \int_0^1 dz'_m z'_m^{(m-2)-(m-1)\epsilon} (1-z'_m)^{1-2\epsilon} \dots \int_0^1 dz'_2 z'_2^{-\epsilon} (1-z'_2)^{1-2\epsilon} \quad (2.40)$$

The definition of the variables  $z_i, z'_i$  is irrelevant here and can be found in ref. [7].

Equations (2.31, 2.32) imply that the phase space depends on  $(1-x)^{-\epsilon}$  through the two variables

$$k'^2 \propto Q^2(1-x) \quad (2.41)$$

$$\frac{(s - q^2)^2}{q^2} \propto Q^2(1-x)^2, \quad (2.42)$$

where the coefficients of proportionality are dimensionless and  $x$ -independent. By explicitly combining the two-body phase space eq. (2.35) and the phase spaces for soft radiation eq. (2.37)

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<sup>1</sup>In the case of deep-inelastic scattering, in ref. [7] one of the outgoing parton momenta ( $k'_{m+1}$ , say) was identified with the momentum of the outgoing quark and called  $p'$ , hence eq. (2.38) is obtained from the corresponding result in ref. [7] by the replacement  $p' \rightarrow k'_{m+1}$ .

and for collinear radiation eq. (2.38) we get

$$\begin{aligned}
d\phi_{n+m+2}(p_1 + p_2; p_\gamma, k_1, \dots, k_n, k'_1, \dots, k'_{m+1}) &= (Q^2)^{n+m-(n+m+1)\epsilon} \frac{dp_\perp}{p_\perp} \frac{(1-x)^{2n+m-(2n+m)\epsilon}}{\sqrt{1-x}} \\
&\times 2^{-m+m\epsilon} \frac{(16\pi)^{-1+\epsilon}}{\Gamma(1-\epsilon)} \left[ \frac{N(\epsilon)}{2\pi} \right]^{n+m} d\eta_\gamma d\Omega^{(n)}(\epsilon) d\Omega'^{(m)}(\epsilon) \\
&\times \int_0^1 du \frac{u^{m-m\epsilon}(1-u)^{2n-1-2n\epsilon}}{\sqrt{u}} \int_0^1 dv \frac{v^{m-1-m\epsilon}}{\sqrt{1-v}} [\delta(\eta_\gamma - \eta_+) + \delta(\eta_\gamma - \eta_-)].
\end{aligned} \tag{2.43}$$

In the limiting cases  $n = 0$  and  $m = 0$  we have

$$d\phi_1(p_1 + p_2; q) = 2\pi\delta(s - q^2) = \frac{2\pi}{Q^2(1-x)} \delta(1-u) \tag{2.44}$$

$$d\phi_1(k'; p') = 2\pi\delta(k'^2) = \frac{4\pi}{Q^2 u(1-x)} \delta(v); \tag{2.45}$$

the corresponding expressions for the phase space are therefore obtained by simply replacing

$$(1-u)^{-1} d\Omega^{(n)}(\epsilon) \rightarrow \delta(1-u); \quad v^{-1} d\Omega'^{(m)}(\epsilon) \rightarrow \delta(v) \tag{2.46}$$

in eq. (2.43) for  $n = 0$ ,  $m = 0$  respectively.

The logarithmic dependence of the four-dimensional cross section on  $1-x$  is due to interference between powers of  $(1-x)^{-\epsilon}$  and  $\frac{1}{\epsilon}$  poles in the  $d$ -dimensional cross section. Hence, we must classify the dependence of the cross section on powers of  $(1-x)^{-\epsilon}$ . We have established that in the phase space each real emission produces a factor of  $[Q^2(1-x)^2]^{1-\epsilon}$  if the emission is soft and a factor of  $[Q^2(1-x)]^{1-\epsilon}$  if the emission is collinear. The squared amplitude can only depend on  $(1-x)^{-\epsilon}$  because of loop integrations. This dependence for a generic proper Feynman diagram  $G$  will in general appear [7] through a coefficient

$$A_G(P_E) = [D_G(P_E)]^{dL/2-I}, \tag{2.47}$$

where  $L$  and  $I$  are respectively the number of loops and internal lines in  $G$ , and  $D_G(P_E)$  is a linear combination of all scalar products  $P_E$  of external momenta. In the soft limit it is easy to see, by manipulations analogous to eq. (2.17), that all scalar products which vanish as  $x \rightarrow 1$  are either proportional to  $Q^2(1-x)$  or to  $Q^2(1-x)^2$ . Equation (2.47) then implies that each loop integration can carry at most a factor of  $[Q^2(1-x)^2]^{-\epsilon}$  or  $[Q^2(1-x)]^{-\epsilon}$ .

This then proves that the perturbative expansion of the bare coefficient function takes the form

$$C^{(0)}(x, Q^2, \alpha_0, \epsilon) = \alpha\alpha_0(Q^2)^{-\epsilon} \sum_{l=0}^{\infty} \alpha_l^l C_l^{(0)}(x, Q^2, \epsilon) \tag{2.48}$$

$$C_l^{(0)}(x, Q^2, \alpha_0, \epsilon) = \frac{(Q^2)^{-l\epsilon}}{\Gamma(1/2)\sqrt{1-x}} \sum_{k=0}^l \sum_{k'=0}^{l-k} C_{lkk'}^{(0)}(\epsilon) (1-x)^{-2k\epsilon-k'\epsilon}, \tag{2.49}$$

where the coefficients  $C_{lkk'}^{(0)}$  have poles in  $\epsilon = 0$  up to order  $2l$ , and the factor  $1/\Gamma(1/2)$  was introduced for later convenience. Terms with  $k + k' < l$  at order  $\alpha_s^l$  are present in general because of loops. The Mellin transform of eq. (2.48) can be performed using

$$\int_0^1 dx x^{N-1} (1-x)^{-1/2-2k\epsilon-k'\epsilon} = \frac{\Gamma(1/2)}{\sqrt{N}} N^{2k\epsilon} N^{k'\epsilon} + O\left(\frac{1}{N}\right), \quad (2.50)$$

with the result

$$\begin{aligned} C^{(0)}(N, Q^2, \alpha_0, \epsilon) \\ = \frac{\alpha \alpha_0 (Q^2)^{-\epsilon}}{\sqrt{N}} \sum_{l=0}^{\infty} \sum_{k=0}^l \sum_{k'=0}^{l-k} C_{lkk'}^{(0)}(\epsilon) \left[ \left( \frac{Q^2}{N^2} \right)^{-\epsilon} \alpha_0 \right]^k \left[ \left( \frac{Q^2}{N} \right)^{-\epsilon} \alpha_0 \right]^{k'} [(Q^2)^{-\epsilon} \alpha_0]^{l-k-k'} + O\left(\frac{1}{N}\right). \end{aligned} \quad (2.51)$$

### 3 Resummation from renormalization group improvement

Equation (2.51) shows that indeed as  $N \rightarrow \infty$ , up to  $\frac{1}{N}$  corrections, the coefficient function depends on  $N$  through the two dimensionful variables  $\frac{Q^2}{N^2}$  and  $\frac{Q^2}{N}$ . The argument henceforth is a rerun of that of ref. [7], in this somewhat more general situation. The argument is based on the observation that, because of collinear factorization, the physical anomalous dimension

$$\gamma(N, \alpha_s(Q^2)) = Q^2 \frac{\partial}{\partial Q^2} \ln C(N, Q^2/\mu^2, \alpha_s(\mu^2)) \quad (3.1)$$

is renormalization-group invariant and finite when expressed in terms of the renormalized coupling  $\alpha_s(\mu^2)$ , related to  $\alpha_0$  by

$$\alpha_0(\mu^2, \alpha_s(\mu^2)) = \mu^{2\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon), \quad (3.2)$$

where  $Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon)$  is a power series in  $\alpha_s(\mu^2)$ . Because  $\alpha_0$  is manifestly independent of  $\mu^2$ , eq. (3.2) implies that the dimensionless combination  $(Q^2)^{-\epsilon} \alpha_0(\alpha_s(\mu^2), \mu^2)$  can depend on  $Q^2$  only through  $\alpha_s(Q^2)$ :

$$(Q^2)^{-\epsilon} \alpha_0(\alpha_s(\mu^2), \mu^2) = \alpha_s(Q^2) Z^{(\alpha_s)}(\alpha_s(Q^2), \epsilon). \quad (3.3)$$

Using eq. (3.3) in eq. (2.51), the coefficient function and consequently the physical anomalous dimension are seen to be given by a power series in  $\alpha_s(Q^2)$ ,  $\alpha_s(Q^2/N)$  and  $\alpha_s(Q^2/N^2)$ :

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \gamma_{mnp}^R(\epsilon) \alpha_s^m(Q^2) \alpha_s^n(Q^2/N^2) \alpha_s^p(Q^2/N). \quad (3.4)$$

Even though the anomalous dimension is finite as  $\epsilon \rightarrow 0$ , the individual terms in the expansion eq. (3.4) are not separately finite. However, if we separate  $N$ -dependent and  $N$ -independent terms in eq. (3.4):

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(N, \alpha_s(Q^2), \epsilon), \quad (3.5)$$

we note that the two functions

$$\gamma^{(c)}(\alpha_s(Q^2), \epsilon) \equiv \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(1, \alpha_s(Q^2), \epsilon) \quad (3.6)$$

$$\gamma^{(l)}(N, \alpha_s(Q^2), \epsilon) \equiv \hat{\gamma}^{(l)}(N, \alpha_s(Q^2), \epsilon) - \hat{\gamma}^{(l)}(1, \alpha_s(Q^2), \epsilon) \quad (3.7)$$

must be separately finite, because

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(c)}(\alpha_s(Q^2), \epsilon) + \gamma^{(l)}(N, \alpha_s(Q^2), \epsilon), \quad (3.8)$$

is finite for all  $N$ , and  $\gamma^{(l)}$  vanishes for  $N = 1$ .

We can rewrite conveniently

$$\gamma^{(l)}(N, \alpha_s(Q^2), \epsilon) = \int_1^N \frac{dn}{n} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon), \quad (3.9)$$

where

$$g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon) = n \frac{\partial}{\partial n} \hat{\gamma}^{(l)}(n, \alpha_s(Q^2), \epsilon). \quad (3.10)$$

is a Taylor series in its arguments whose coefficients remain finite as  $\epsilon \rightarrow 0$ . In four dimension we have thus

$$\begin{aligned} \gamma(N, \alpha_s(Q^2)) &= \gamma^{(l)}(N, \alpha_s(Q^2), 0) + \gamma^{(c)}(N, \alpha_s(Q^2), 0) + O\left(\frac{1}{N}\right) \\ &= \gamma^{(l)}(N, \alpha_s(Q^2), 0) + O(N^0) \\ &= \int_1^N \frac{dn}{n} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) + O(N^0), \end{aligned} \quad (3.11)$$

where  $g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) \equiv \lim_{\epsilon \rightarrow 0} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon)$  is a generic Taylor series of its arguments.

Renormalization group invariance thus implies that the physical anomalous dimension  $\gamma$  eq. (3.1) depends on its three arguments  $Q^2$ ,  $Q^2/N$  and  $Q^2/N^2$  only through  $\alpha_s$ . Clearly, any function of  $Q^2$  and  $N$  can be expressed as a function of  $\alpha_s(Q^2)$  and  $\alpha_s(Q^2/N)$  or  $\alpha_s(Q^2/N^2)$ . The nontrivial statement, which endows eq. (3.11) with predictive power, is that the log derivative of  $\gamma$ ,  $g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n))$  eq. (3.10), is analytic in its three arguments. This immediately implies that when  $\gamma$  is computed at (fixed) order  $\alpha_s^k$ , it is a polynomial in  $\ln \frac{1}{N}$  of  $k$ -th order at most.

In order to discuss the factorization properties of our result we write the function  $g$  as

$$\begin{aligned} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) &= g_1(\alpha_s(Q^2), \alpha_s(Q^2/n)) + g_2(\alpha_s(Q^2), \alpha_s(Q^2/n^2)) \\ &\quad + g_3(\alpha_s(Q^2), \alpha_s(Q^2/n), \alpha_s(Q^2/n^2)) \\ g_1(\alpha_s(Q^2), \alpha_s(Q^2/n)) &= \sum_{m=0}^{\infty} \sum_{p=1}^{\infty} g_{m0p} \alpha_s^m(Q^2) \alpha_s^p(Q^2/n) \\ g_2(\alpha_s(Q^2), \alpha_s(Q^2/n^2)) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} g_{mn0} \alpha_s^m(Q^2) \alpha_s^n(Q^2/n^2) \\ g_3(\alpha_s(Q^2), \alpha_s(Q^2/n), \alpha_s(Q^2/n^2)) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} g_{mnp} \alpha_s^m(Q^2) \alpha_s^n(Q^2/n^2) \alpha_s^p(Q^2/n). \end{aligned} \quad (3.12)$$

The dependence on the resummation variables is fully factorized if the bare coefficient functions has the factorized structure

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = C^{(0,c)}(Q^2, \alpha_0, \epsilon) C_1^{(0,l)}(Q^2/N, \alpha_0, \epsilon) C_2^{(0,l)}(Q^2/N^2, \alpha_0, \epsilon). \quad (3.13)$$

This is argued to be the case in the approach of refs. [8, 10]. If this happens, the resummed anomalous dimension is given by eq. (3.11) with all  $g_{mnp} = 0$  except  $g_{0n0}, g_{00p}$ :

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n} g_1(0, \alpha_s(Q^2/n)) + \int_1^N \frac{dn}{n} g_2(0, \alpha_s(Q^2/n^2)). \quad (3.14)$$

Note that because the coefficient function depends on the parton subprocess (compare eq. (2.3)) the factorization eq. (3.13) applies to the coefficient function corresponding to each subprocess, and the decomposition eq. (3.14) to the physical anomalous dimension computed from each of these coefficient functions.

A weaker form of factorization is obtained assuming that in the soft limit the  $N$ -dependent and  $N$ -independent parts of the coefficient function factorize:

$$C^{(0)}(N, Q^2, \epsilon) = C^{(0,c)}(Q^2, \alpha_0, \epsilon) C^{(0,l)}(Q^2/N^2, Q^2/N, \alpha_0, \epsilon). \quad (3.15)$$

This condition turns out to be satisfied [7] in Drell-Yan and deep-inelastic scattering to order  $\alpha_s^2$ . It holds in QED to all orders [13] as a consequence of the fact that each emission in the soft limit can be described by a universal (eikonal) factors, independent of the underlying diagram. This eikonal structure of Sudakov radiation has been argued in refs. [2, 10] to apply also to QCD. If the factorized form eq. (3.15) holds, the coefficients  $g_{mnp}$  eq. (3.12) vanish for all  $m \neq 0$ , and the physical anomalous dimension takes the form

$$\begin{aligned} \gamma(N, \alpha_s(Q^2)) &= \int_1^N \frac{dn}{n} g_1(0, \alpha_s(Q^2/n)) + \int_1^N \frac{dn}{n} g_2(0, \alpha_s(Q^2/n^2)) \\ &\quad + \int_1^N \frac{dn}{n} g_3(0, \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)). \end{aligned} \quad (3.16)$$

It is interesting to observe that in the approach of refs. [8, 10] for processes where more than one colour structure contributes to the cross-section, the factorization eq. (3.13) of the coefficient function is argued to take place separately for each colour structure. This means that in such case the exponentiation takes place for each colour structure independently, i.e. the resummed cross section for each parton subprocess is in turn expressed as a sum of factorized terms of the form of eq. (3.13). This happens for instance in the case of heavy quark production [10, 12]. In prompt photon production different colour structures appear for the gluon-gluon subprocess which starts at next-to-next-to-leading order, hence their separated exponentiation would be relevant for next-to-next-to-leading log resummed results.

When several colour structures contribute to a given parton subprocess, the coefficients of the perturbative expansion eq. (2.51) for that process take the form

$$C_{lkk'}^{(0)}(\epsilon) = C_{lkk'}^{(0)1}(\epsilon) + C_{lkk'}^{(0)8}(\epsilon), \quad (3.17)$$

(assuming for definiteness that a colour singlet and octet contribution are present) so that the coefficient function can be written as a sum  $C^{(0)} = C_1^{(0)} + C_8^{(0)}$ . The argument which leads from eq. (2.51) to the resummed result eq. (3.11) then implies that exponentiation takes place for each colour structure independently if and only if  $\gamma_1 \equiv \partial \ln C_1^{(0)} / \partial \ln Q^2$  and  $\gamma_8 \equiv \partial \ln C_8^{(0)} / \partial \ln Q^2$  are separately finite.

This, however, is clearly a more restrictive assumption than that under which we have derived the result eq. (3.11), namely that the full anomalous dimension  $\gamma$  is finite. It follows that exponentiation of each colour structure must be a special case of our result. However, this can only be true if the coefficients  $g_{ijk}$  of the expansion eq. (3.12) of the physical anomalous dimension satisfy suitable relations. In particular, at the leading log level, it is easy to see that exponentiation of each colour structure is compatible with exponentiation of their sum only if the leading order coefficients are the same for the given colour structures:  $g_{001}^1 = g_{001}^8$  and  $g_{010}^1 = g_{010}^8$ . This is indeed the case for heavy quark production (where  $g_{001} = 0$ ).

Note that, however, if the factorization holds for each colour structure separately it will not apply to the sum of colour structures. For instance, the weaker form of factorization eq. (3.15) requires that  $C_{lkk'}^{(0)}(\epsilon) = F_{k+k'}(\epsilon)G_{l-k-k'}(\epsilon)$ , but

$$F_{k+k'}^1(\epsilon)G_{l-k-k'}^1(\epsilon) + F_{k+k'}^8(\epsilon)G_{l-k-k'}^8(\epsilon) \neq F_{k+k'}(\epsilon)G_{l-k-k'}(\epsilon). \quad (3.18)$$

Hence, our result eq. (3.11) for the sum of colour structures is more general than the separate exponentiation of individual colour structures, but it leads to results which have weaker factorization properties.

The resummation coefficients  $g_{mnp}$  can be determined by comparing the expansion of the resummed anomalous dimension  $\gamma$  in powers of  $\alpha_s(Q^2)$  with a fixed-order calculation:

$$\gamma_{\text{FO}}(N, \alpha_s) = \sum_{i=1}^{k_{\min}} \alpha_s^i \sum_{j=1}^i \gamma_j^i \ln^j \frac{1}{N} + O(\alpha_s^{k_{\min}+1}) + O(N^0), \quad (3.19)$$

where  $\gamma_{\text{FO}}(N, \alpha_s)$  is the physical anomalous dimension for the same individual partonic subprocess (recall eq. (2.3)). Clearly, if the more restrictive factorized forms eq. (3.13) or eq. (3.15) hold, a smaller number of coefficients determines the resummed result, and thus a lower fixed-order calculation is sufficient to predict higher-order logarithmic terms than if the more general eq. (3.12) is used. Conversely, a higher fixed-order calculation can be used to verify if the strong factorization eq. (3.13) holds as advocated in refs. [8, 10], or whether one must use the less predictive but more general result eq. (3.12) that we have derived.

Once the resummed physical anomalous dimension has been determined, the resummed cross section can be obtained from it using eq. (2.12), with a factorization scheme choice which specifies the way it is split into its two components eq. (2.11). Commonly used choices are the physical scheme choice, in which  $C = 1$  so  $\gamma = \gamma^{\text{AP}}$ , or the  $\overline{\text{MS}}$  scheme, in which the unresummed and resummed forms of the anomalous dimensions  $\gamma^{\text{AP}}$  coincide. An explicit construction of the relation between physical anomalous dimension and resummed cross section, and the matching between resummed and unresummed results, can be found in Section 6 of ref. [7].

## 4 The structure of resummed results

We determine the predictive power of each resummed result by means of the following strategy. First, we assume that the coefficients  $g_{mnp}$  needed for  $N^{k-2}$ LL resummation have already been determined. Next, we identify the coefficients that are needed to extend the accuracy to  $N^{k-1}$ LL, and we write a system of equations that fix them in terms of the known coefficients, and of the  $\gamma_j^i$  of the fixed-order expansion. The rank of this system of equations determines the minimum order  $k_{\min}$  in  $\alpha_s$  of a fixed order computation which is needed to fix the  $N^{k-1}$ LL resummation. This means that at any higher fixed order  $f > k_{\min}$ , the coefficients of all powers of  $\left(\ln \frac{1}{N}\right)^n$  with  $k < n \leq f$  are then predicted by the resummed formula.

The general structure of the anomalous dimension resummed to  $N^{k-1}$ LL accuracy is

$$\gamma(N, \alpha_s(Q^2)) = \sum_{p=1}^k \gamma_p(N, \alpha_s(Q^2)) \quad (4.1)$$

$$\gamma_p(N, \alpha_s(Q^2)) = \sum_{i=0}^{p-1} \sum_{j=0}^{p-i} g_{ijp-i-j} \alpha_s^i(Q^2) \int_1^N \frac{dn}{n} \alpha_s^j(Q^2/n^2) \alpha_s^{p-i-j}(Q^2/n). \quad (4.2)$$

At the  $N^{k-1}$ LL order, in each term  $\gamma_p(N, \alpha_s(Q^2))$  the coupling constant  $\alpha_s(Q^2/n^a)$  can be expanded in powers of  $\alpha_s(Q^2)$  using the  $N^{k-p}$ LL solution of the renormalization group equation

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 + \dots, \quad (4.3)$$

because subsequent orders would lead to  $N^k$ LL contributions to  $\gamma$ . In particular, the leading log expression

$$\alpha_s(Q^2/n^a) = \frac{\alpha_s(Q^2)}{1 + a\alpha_s(Q^2)\beta_0 \ln \frac{1}{n}} \quad (4.4)$$

is sufficient for the computation of  $\gamma_k(N, \alpha_s(Q^2))$ . With  $\alpha_s(Q^2/n^a)$  given by eq. (4.4) one gets

$$\int_1^N \frac{dn}{n} \alpha_s^i(Q^2/n^2) \alpha_s^j(Q^2/n) = \sum_{m=0}^{\infty} C_m^{(i,j)} \beta_0^m \alpha_s(Q^2)^{i+j+m} \ln^{m+1} \frac{1}{N} \quad (4.5)$$

$$C_m^{(i,j)} = \frac{(-1)^{m+1}}{m+1} \sum_{l=0}^m 2^l \binom{l+i-1}{i-1} \binom{m-l+j-1}{j-1} \quad (4.6)$$

(note that  $\binom{n}{-1} = 1$  for  $n = -1$  and 0 otherwise), and therefore

$$\gamma_k(N, \alpha_s(Q^2)) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-i} g_{ijk-i-j} \sum_{m=0}^{\infty} C_m^{(j,k-i-j)} \beta_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \quad (4.7)$$

Let us consider first the leading log case,  $k = 1$ . In this case,  $g$  is linear in  $\alpha_s$  and therefore eq. (3.14) always holds, i.e. there is no distinction between factorized and unfactorized results. The anomalous dimension has the form

$$\gamma_1(N, \alpha_s(Q^2)) = \sum_{m=0}^{\infty} (g_{001} + 2^m g_{010}) \frac{(-1)^{m+1}}{m+1} \beta_0^m \alpha_s^{m+1}(Q^2) \ln^{m+1} \frac{1}{N} + \text{NLL}. \quad (4.8)$$

Comparing with the fixed order expansion, we find

$$m = 0 : \quad \gamma_1^1 = -(g_{001} + g_{010}) \quad (4.9)$$

$$m = 1 : \quad \gamma_2^2 = \frac{\beta_0}{2} (g_{001} + 2g_{010}). \quad (4.10)$$

These two conditions determine  $g_{001}$  and  $g_{010}$ : leading-log resummation requires computing  $\gamma$  to order  $\alpha_s^2$ . Note that for DIS and Drell-Yan an  $O(\alpha_s)$  computation is instead sufficient [7], because only one scale is present and thus only one coefficient has to be determined.

Let us now examine higher logarithmic orders, by discussing the various factorizations in turn. Assume first the validity of the most restrictive result eq. (3.14), where the dependence on the three scales  $Q^2$ ,  $Q^2/N$  and  $Q^2/N^2$  is fully factorized. In such case, the anomalous dimension is just the sum of a function of  $\alpha_s(Q^2/N)$  and a function of  $\alpha_s(Q^2/N^2)$ , and all coefficients  $g_{mnp}$  vanish except  $g_{00p}$ ,  $g_{0p0}$ . In this case,  $N_k = 2k$  coefficients are required for the  $N^{k-1}$ LL resummation. According to the strategy outlined above, we now assume that the coefficients  $g_{00p}$ ,  $g_{0p0}$  with  $p \leq k-1$ , relevant for  $N^{k-2}$ LL resummation, have already been determined. The two extra coefficients  $g_{00k}$ ,  $g_{0k0}$  appear in  $\gamma_k(N, \alpha_s)$ , whose explicit form is given by eq. (4.7) with only  $g_{00k}$ ,  $g_{0k0}$  different from zero:

$$\gamma_k(N, \alpha_s(Q^2)) = \sum_{m=0}^{\infty} (C_m^{(0,k)} g_{00k} + C_m^{(k,0)} g_{0k0}) \beta_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1} \frac{1}{N}. \quad (4.11)$$

Other terms of order  $\alpha_s^{k+m} \ln^{m+1}(1/N)$  are generated by expanding the coupling  $\alpha_s(Q^2/n^a)$  in  $\gamma_1, \dots, \gamma_{k-1}$  up to  $N^{k-1}$ LL; however, these are fully determined by the known coefficients  $g_{00i}$ ,  $g_{0i0}$ ,  $i \leq k-1$  (and by the coefficients of the beta function  $\beta_0, \dots, \beta_{k-1}$ ). Equating terms of order  $\alpha_s^{k+m} \ln^{m+1}(1/N)$  in the fixed-order and resummed expressions of  $\gamma$  we get the set of equations

$$m = 0 : \quad \gamma_1^k = -(g_{00k} + g_{0k0}) + F_1^{(k)}(g_{00i}, g_{0i0}; 1 \leq i \leq k-1) \quad (4.12)$$

$$m = 1 : \quad \gamma_2^{k+1} = (g_{00k} + 2g_{0k0}) \frac{k\beta_0}{2} + F_2^{(k)}(g_{00i}, g_{0i0}; 1 \leq i \leq k-1) \quad (4.13)$$

$\dots$ ,

where  $F_j^{(i)}$  are known functions of the coefficients  $g$  which we have assumed to be already known.

Hence, the two terms  $m = 0, 1$  provide two independent conditions that fix  $g_{00k}$  and  $g_{0k0}$  in terms of  $g_{00i}$ ,  $g_{0i0}$ ,  $i \leq k-1$ . The same procedure can be repeated for  $p = k-1, k-2, \dots, 1$ ; at each step,  $g_{00p}$  and  $g_{0p0}$  are computed as functions of  $\gamma_1^p, \gamma_2^{p+1}$  and  $g_{00i}, g_{0i0}$ ,  $0 \leq i \leq p-1$ . We conclude that in the case of eq. (3.14) the coefficients  $g_{00i}, g_{0i0}; 0 \leq i \leq k$ , relevant for  $N^{k-1}$ LL resummation, are obtained from the fixed-order expansion of  $\gamma$  up to order  $k_{\min} = k+1$  (corresponding to  $m = 1$  in eq. (4.11)). This means that even though at each extra logarithmic order two new coefficients appear, a single extra fixed order in  $\alpha_s$  is sufficient to determine both of them.

Consider for example the next-to-leading log resummation. In our approach, this requires the calculation of the physical anomalous dimension up to order  $\alpha_s^3$ . Explicitly,

$$\gamma_1(N, \alpha_s(Q^2)) = -(g_{001} + g_{010}) \alpha_s(Q^2) \ln \frac{1}{N}$$

$$\begin{aligned}
& + \frac{\beta_0}{2}(g_{001} + 2g_{010}) \alpha_s^2(Q^2) \ln^2 \frac{1}{N} - \frac{\beta_1}{\beta_0}(g_{001} + g_{010}) \alpha_s^2(Q^2) \ln \frac{1}{N} \\
& - \frac{\beta_0^2}{3}(g_{001} + 4g_{010}) \alpha_s^3(Q^2) \ln^3 \frac{1}{N} + \frac{3\beta_1}{2}(g_{001} + 2g_{010}) \alpha_s^3(Q^2) \ln^2 \frac{1}{N} \quad (4.14)
\end{aligned}$$

$$\gamma_2(N, \alpha_s(Q^2)) = -(g_{002} + g_{020}) \alpha_s^2(Q^2) \ln \frac{1}{N} + \beta_0(g_{002} + 2g_{020}) \alpha_s^3(Q^2) \ln^2 \frac{1}{N}. \quad (4.15)$$

The leading log coefficients are fixed by the linear and quadratic  $\ln \frac{1}{N}$  terms in  $\gamma_1$ , eqs. (4.9,4.10), and the NLL coefficients by the linear and quadratic  $\ln \frac{1}{N}$  terms in  $\gamma_2$ :

$$g_{002} + g_{020} = -\gamma_1^2 - \frac{\beta_1}{\beta_0}(g_{001} + g_{010}) \quad (4.16)$$

$$g_{002} + 2g_{020} = \frac{1}{\beta_0}\gamma_2^3 - \frac{3}{2}\frac{\beta_1}{\beta_0}(g_{001} + 2g_{010}). \quad (4.17)$$

All other logarithmically enhanced contributions to the order  $\alpha_s^3$  anomalous dimension are predicted, except the  $\alpha_s^3 \ln \frac{1}{N}$  which depends on the N<sup>2</sup>LL coefficients. The same pattern continues at higher orders.

It is interesting to observe that even assuming the fully factorized form of the coefficient function eq. (3.13), our results are still less restrictive and thus less predictive than those of refs. [8, 10], where one less fixed order is required to determine the resummed result. For instance, at NLL, using the notation of ref. [10], the resummation coefficients are given by

$$g_{010} = -\frac{A_a^{(1)} + A_b^{(1)} - A_d^{(1)}}{\pi} \quad (4.18)$$

$$g_{001} = -\frac{A_d^{(1)}}{\pi} \quad (4.19)$$

$$g_{020} = -\left[ \frac{A_a^{(2)} + A_b^{(2)} - A_d^{(2)}}{\pi^2} - \frac{\beta_0(A_a^{(1)} + A_b^{(1)} - A_d^{(1)})}{\pi} \ln 2 - 2\gamma_E \beta_0 \frac{A_a^{(1)} + A_b^{(1)} - A_d^{(1)}}{\pi} \right] \quad (4.20)$$

$$g_{002} = -\left[ \frac{A_d^{(2)}}{\pi^2} - \frac{\beta_0 B_d^{(1)}}{2\pi} - \gamma_E \beta_0 \frac{A_d^{(1)}}{\pi} \right], \quad (4.21)$$

where  $A_a^{(i)}$  is the coefficient of  $\ln(1/N)$  in the Mellin transform of the  $P_{aa}$  Altarelli-Parisi splitting function at order  $\alpha_s^i$ ,  $\gamma_E$  is the Euler constant, and  $B_d^{(1)}$  is a constant to be determined from the comparison with the fixed-order calculation. In eq. (4.18-4.21)  $a, b$  are the incoming partons (on which  $C$  implicitly depends), and  $d$  is the outgoing parton in the leading order process when incoming partons  $a$  and  $b$  (which is uniquely determined by  $a$  and  $b$ ).

Thus, in this approach  $g_{0i0}$  is entirely determined in terms of the  $O(\alpha_s^i)$  coefficient of the  $\ln(1/N)$  term in the anomalous dimension, and only  $g_{00i}$  must be determined by comparison to the fixed-order calculation: at the LL level, the resummed result is obtained from the knowledge of  $\gamma_{\text{FO}}$  to order  $\alpha_s$ , at the NLL level to order  $\alpha_s^2$  and so on. This means that to LL, the coefficient  $\gamma_2^2$  eq. (4.10) is in fact predicted by eqs. (4.18,4.19) in terms of the coefficients of the Altarelli-Parisi splitting functions. This prediction is borne out by the explicit  $O(\alpha_s^2)$  calculation of the

prompt photon production cross section [14]. At the NLL level, the coefficient  $\gamma_2^3$  of  $\alpha_s^3 \ln^2(1/N)$ , is predicted:

$$\begin{aligned} \gamma_2^3 = & -\beta_0 \left[ \frac{2(A_a^{(2)} + A_b^{(2)}) - A_d^{(2)}}{\pi^2} - \beta_0(2 \ln 2 + 4\gamma_E) \frac{A_a^{(1)} + A_b^{(1)}}{\pi} \right. \\ & \left. + \beta_0(2 \ln 2 + 3\gamma_E) \frac{A_d^{(1)}}{\pi} - \frac{\beta_0 B_d^{(1)}}{2\pi} \right] - \beta_1 \frac{3}{2} \left[ \frac{2(A_a^{(1)} + A_b^{(1)}) - A_d^{(1)}}{\pi} \right]. \end{aligned} \quad (4.22)$$

The correctness of this result could be tested by an order  $\alpha_s^3$  calculation. If it were to fail, the more general resummation formula with  $g_{020}$  determined by eq. (4.17) should be used, or one of the resummations which do not assume the factorization eq. (3.13).

Assume now that the weaker factorization eq. (3.15) applies. In this case, only  $g_{0nq}; 1 \leq n + q \leq k$  are nonzero. This amounts to keeping only the term  $i = 0$  in the general expression eq. (4.2) for  $\gamma_p(N, \alpha_s(Q^2))$ . The total number of coefficients at  $N^{k-1}\text{LL}$  is

$$N_k = \sum_{p=1}^k (p+1) = \frac{k(k+3)}{2}. \quad (4.23)$$

In order to improve the accuracy from  $N^{k-2}\text{LL}$  to  $N^{k-1}\text{LL}$ ,  $k+1$  new coefficients are needed, namely  $g_{0ik-i}$  for  $i = 0, \dots, k$ . As in the previous case, we isolate the  $N^{k-1}\text{LL}$  terms that depend on the new coefficients. All such terms are contained in  $\gamma_k$ , which in this case is given by the term  $i = 0$  in eq. (4.7):

$$\gamma_k(N, \alpha_s(Q^2)) = \sum_{j=0}^k g_{0jk-j} \sum_{m=0}^{\infty} C_m^{(j,k-j)} \beta_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \quad (4.24)$$

The first  $k+1$  terms in the sum over  $m$  provide the set of conditions

$$m = 0 : \quad \gamma_1^k = \sum_{j=0}^k g_{0jk-j} C_0^{(j,k-j)} + G_1^{(k)}(g_{0jk-j}; 0 \leq j \leq k-1) \quad (4.25)$$

$$m = 1 : \quad \gamma_2^{k+1} = \sum_{j=0}^k g_{0jk-j} C_1^{(j,k-j)} \beta_0 + G_2^{(k)}(g_{0jk-j}; 0 \leq j \leq k-1) \quad (4.26)$$

...

$$m = k : \quad \gamma_{k+1}^{2k} = \sum_{j=0}^k g_{0jk-j} C_k^{(j,k-j)} \beta_0^k + G_{k+1}^{(k)}(g_{0jk-j}; 0 \leq j \leq k-1), \quad (4.27)$$

where again  $G_j^{(i)}$  are known functions of the coefficients  $g$  which we have assumed to be already known. Equations (4.25-4.27) are linearly independent (see the Appendix for an explicit proof), and therefore determine  $g_{0ik-i}; 0 \leq i \leq k$  in terms of  $g_{0jk-j}; 0 \leq j \leq k-1$ . It follows that a computation of  $\gamma$  up to order  $k_{\min} = 2k$  is necessary for the determination of the  $\frac{k(k+3)}{2}$  coefficients needed for  $N^{k-1}\text{LL}$  resummation: even though the number of coefficients which must be determined grows quadratically according to eq. (4.23), the required order in  $\alpha_s$  of the computation which determines them grows only linearly.

Note that in fact a similar argument can be applied to the cases of deep-inelastic scattering and Drell-Yan in the absence of factorization, discussed in ref. [7]. In that reference it was observed that  $k(k+1)/2$  coefficients must be known in order to determine the  $N^{k-1}\text{LL}$  result, and thus a fixed  $N^{k(k+1)/2-1}\text{LO}$  computation is sufficient to determine them. However, using the argument given here it is easy to see that in fact a calculation of  $\gamma$  up to fixed order  $k_{\min} = 2k - 1$  only is sufficient to determine all these coefficients. Consider for example the case of deep-inelastic scattering. To  $N^{k-1}\text{LL}$  one finds

$$\gamma(N, \alpha_s(Q^2)) = \sum_{p=1}^k \gamma_p(N, \alpha_s(Q^2)) \quad (4.28)$$

$$\gamma_p(N, \alpha_s(Q^2)) = \sum_{i=0}^{p-1} g_{ip-i} \alpha_s^i(Q^2) \int_1^N \frac{dn}{n} \alpha_s^{p-i}(Q^2/n). \quad (4.29)$$

In order to determine the  $k$  coefficients  $g_{0k}, \dots, g_{k-11}$  that are needed to improve the accuracy from  $N^{k-2}\text{LL}$  to  $N^{k-1}\text{LL}$ , we proceed as in the previous case: the new coefficients appear only in

$$\gamma_k(N, \alpha_s(Q^2)) = \sum_{i=0}^{k-1} g_{ik-i} \sum_{m=0}^{\infty} C_m^{(0,k-i)} \beta_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \quad (4.30)$$

Each term in the sum over  $m$  in eq. (4.30) provides an independent condition on the coefficients  $g_{ij}$  (the linear independence of these conditions is straightforwardly proved in the Appendix). Hence, in order to determine all  $g_{ik-i}; 0 \leq i \leq k-1$  it is sufficient to determine all terms up to  $m = k-1$  in  $\gamma_k$ , i.e. compute  $\gamma$  up to order  $\alpha_s^{2k-1}$ . The same happens in the case of Drell-Yan, which is obtained by replacing  $C_m^{(0,p-i)}$  with  $C_m^{(p-i,0)}$  in eq. (4.30).

Let us now consider the most general case, in which the coefficient function does not satisfy any factorization property. Then, to  $N^{k-1}\text{LL}$ , the anomalous dimension eqs. (4.1,4.2) depends on

$$N_k = \sum_{p=1}^k \frac{p(p+3)}{2} = \frac{k(k+1)(k+5)}{6} \quad (4.31)$$

coefficients overall; out of these, the new ones which must be determined in order to go from  $N^{k-2}\text{LL}$  to  $N^{k-1}\text{LL}$  are the  $k(k+3)/2$  coefficients

$$g_{ijk-i-j}; \quad i = 0, \dots, k-1; \quad j = 0, \dots, k-i. \quad (4.32)$$

These new coefficients are contained in  $\gamma_k$ , now given by its general expression eq. (4.7), and each term with fixed  $m$  in the expansion of  $\gamma_k$  eq. (4.7) provides a new condition on these coefficient. However, these conditions are not linearly independent for any choice of  $m$ : rather, the rank of the matrix which gives the linear combination of coefficients eq. (4.32) to be determined turns out to be  $2k$  (see the Appendix). This means that the  $N^{k-1}\text{LL}$  order resummed result depends only on  $2k$  independent linear combinations of the  $k(k+3)/2$  coefficients eq. (4.32). Because a term with fixed  $m$  in  $\gamma_k$  is of order  $\alpha_s^{k+m}$ , this implies that a computation of the anomalous dimension up to fixed order  $k_{\min} = 3k - 1$  is sufficient for the  $N^{k-1}\text{LL}$  resummation. Note that when going from  $N^{k-1}\text{LL}$  to  $N^k\text{LL}$  and  $\gamma_k$  is now determined at this higher order, in general

	Prompt photon			DIS, DY	
	eq. (3.14)	eq. (3.16)	eq. (3.11)	refs. [2, 3]	ref. [7]
$N_k$	$2k$	$\frac{k(k+3)}{2}$	$\frac{k(k+1)(k+5)}{6}$	$k$	$\frac{k(k+1)}{2}$
$k_{\min}$	$k + 1$	$2k$	$3k - 1$	$k$	$2k - 1$

Table 1: Number of coefficients  $N_k$  and minimum order of the required perturbative calculation  $k_{\min}$  for different versions of the  $N^{k-1}\text{LL}$  resummation.

some new linear combinations of the  $k(k+3)/2$  coefficients eq. (4.32) will appear through terms depending on  $\beta_1$ . Hence, some of the combinations of coefficients that were left undetermined in the  $N^{k-1}\text{LL}$  resummation will now become determined. However, this does not affect the value  $k_{\min}$  of the fixed-order accuracy needed to push the resummed accuracy at one extra order. In conclusion, even in the absence of any factorization, despite the fact that now the number of coefficients which must be determined grows cubically according to eq. (4.31), the required order in  $\alpha_s$  of the computation which determines them grows only linearly.

The number of coefficients  $N_k$  that must be determined at each logarithmic order, and the minimum fixed order which is necessary in order to determine them are summarized in Table 1, according to whether the coefficient function is fully factorized [eq. (3.14)], or has factorized  $N$ -dependent and  $N$ -independent terms [eq. (3.16)], or not factorized at all [eq. (3.11)]. In the approach of refs. [8, 10] the coefficient function is fully factorized, and furthermore some resummation coefficients are related to universal coefficients of Altarelli-Parisi splitting functions, so that  $k_{\min} = k$ . For completeness, we also list in the table the results for DIS and Drell-Yan, according to whether the coefficient function has factorized  $N$ -dependent and  $N$ -independent terms (as in refs. [2, 3]) or no factorization properties (as in ref. [7]). Current fixed-order results support factorization for Drell-Yan and DIS only to the lowest nontrivial order  $O(\alpha_s^2)$ . For prompt-photon production, available results do not allow to test factorization, and test relation of resummation coefficients to Altarelli-Parisi coefficients only to lowest  $O(\alpha_s)$ .

## 5 Conclusion

In this paper, we have presented a generalization to prompt photon production of the approach to Sudakov resummation which was introduced in ref. [7] for deep-inelastic scattering and Drell-Yan production. The advantage of this approach is that it does not rely on factorization of the physical cross section, and in fact it simply follows from general kinematic properties of the phase space. It is interesting to see that this remains true even with the more intricate two-scale kinematics that characterizes prompt photon production in the soft limit, especially in view of the fact that the theoretical status of Sudakov resummation for prompt photon production is rather less satisfactory than for DIS or Drell-Yan. Also, this approach does not require a separate treatment of individual colour structures when more than one colour structure contributes to fixed order results.

The resummation formulae obtained here turn out to be less predictive than previous re-

sults [8, 10]: a higher fixed-order computation is required in order to determine the resummed result. This depends on the fact that here neither specific factorization properties of the cross section in the soft limit is assumed, nor that soft emission satisfies eikonal-like relations which allow one to determine some of the resummation coefficients in terms of universal properties of collinear radiation. Currently, fixed-order results are only available up to  $O(\alpha_s^2)$  for prompt photon production. An order  $\alpha_s^3$  computation is required to check nontrivial properties of the structure of resummation: for example, factorization, whose effects only appear at the next-to-leading log level, can only be tested at  $O(\alpha_s^3)$ . The greater flexibility of the approach presented here would turn out to be necessary if the prediction obtained using the more restrictive resummation of refs. [8, 10] were to fail at order  $\alpha_s^3$ .

## A Appendix

In this Appendix, we prove some properties of matrices built from the coefficients eq. (4.6), which appear in the perturbative expansion of the highest order contribution  $\gamma_k$  to the resummed anomalous dimension eq. (4.1,4.2).

1. The  $k \times k$  matrix

$$A_{mi}^{(k)} = C_m^{(0,k-i)}; \quad 0 \leq m \leq k-1, \quad 0 \leq i \leq k-1 \quad (\text{A.1})$$

is non-singular.

*Proof:* From eq. (4.6) we see that  $A_{mi}^{(k)}$  is a degree- $m$  polynomial in  $i$ :

$$A_{mi}^{(k)} = \frac{(-1)^{m+1}}{(m+1)!} (m+k-i-1) \times \dots \times (k-i) = \sum_{l=0}^m A_l i^l. \quad (\text{A.2})$$

It follows that a generic linear combination of the rows of  $A^{(k)}$

$$\sum_{m=0}^{k-1} x_m A_{mi}^{(k)} = \sum_{m=0}^{k-1} x_m \sum_{l=0}^m A_l i^l = \sum_{l=0}^{k-1} A_l i^l \sum_{m=0}^l x_m \quad (\text{A.3})$$

can only vanish if  $x_m = 0$  for all  $m$ .

It follows that each term in the sum over  $m$  in eq. (4.30) provides a linearly independent condition on the coefficients  $g_{ik-i}$ ,  $0 \leq i \leq k-1$ .

2. The  $k \times k$  matrix

$$A_{mi}^{(k)} = C_m^{(k-i,0)}; \quad 0 \leq m \leq k-1, \quad 0 \leq i \leq k-1 \quad (\text{A.4})$$

is non-singular.

*Proof:* This statement follows immediately from the previous one, because it is easy to show that

$$C_m^{(k-i,0)} = 2^m C_m^{(0,k-i)}. \quad (\text{A.5})$$

3. The  $(k+1) \times (k+1)$  matrix

$$B_{mj}^{(k)} = C_m^{(j,k-j)}; \quad 0 \leq m \leq k, \quad 0 \leq j \leq k \quad (\text{A.6})$$

is non-singular.

*Proof:* This statement can be proved by induction on  $k$ . For  $k = 1$  we have

$$B^{(1)} = \begin{pmatrix} -1 & -1 \\ 1/2 & 1 \end{pmatrix} \quad (\text{A.7})$$

which is manifestly non-singular. We now assume that  $B^{(k-1)}$  is non-singular, and we consider a linear combination of the columns of  $B^{(k)}$ :

$$\sum_{j=0}^k x_j B_{mj}^{(k)} = (x_0 + 2^m x_k) C_m^{(0,k)} + \sum_{j=1}^{k-1} x_j C_m^{(j,k-j)}, \quad (\text{A.8})$$

where we have used eq. (A.5). For  $i, j \geq 1$  the following identity holds:

$$C_m^{(i,j)} = 2C_m^{(i,j-1)} - C_m^{(i-1,j)}. \quad (\text{A.9})$$

Equation (A.9) can be verified directly, using the standard properties of the binomial coefficients

$$\binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k}; \quad \binom{n}{k} = 0 \text{ for } n < k. \quad (\text{A.10})$$

Using eq. (A.9) we get

$$\begin{aligned} \sum_{j=0}^k x_j B_{mj}^{(k)} &= (x_0 + 2^m x_k) C_m^{(0,k)} + 2 \sum_{j=1}^{k-1} x_j C_m^{(j,k-j-1)} - \sum_{j=1}^{k-1} x_j C_m^{(j-1,k-j)} \\ &= (x_0 + 2^m x_k) C_m^{(0,k)} + 2 \sum_{j=1}^{k-1} x_j C_m^{(j,k-1-j)} - \sum_{j=0}^{k-2} x_{j+1} C_m^{(j,k-1-j)} \\ &= (x_0 + 2^m x_k) C_m^{(0,k)} + \sum_{j=0}^{k-1} \bar{x}_j B_{mj}^{(k-1)}, \end{aligned} \quad (\text{A.11})$$

where

$$\begin{aligned} \bar{x}_0 &= -x_1 \\ \bar{x}_j &= 2x_j - x_{j+1}; \quad 1 \leq j \leq k-2 \\ \bar{x}_{k-1} &= 2x_{k-1}. \end{aligned} \quad (\text{A.12})$$

The linear combination in eq. (A.11) can only vanish if the two terms are separately zero, since  $C_m^{(0,k)}$  is a degree- $(k-1)$  polynomial in  $m$ , while  $B_{mj}^{(k-1)}$  is at most of degree  $k-2$ . Hence, for eq. (A.11) to vanish, it must be

$$x_0 + 2^m x_k = 0 \quad (\text{A.13})$$

$$\sum_{j=0}^{k-1} \bar{x}_j B_{mj}^{(k-1)} = 0, \quad (\text{A.14})$$

and therefore

$$x_0 = 0; \quad x_k = 0; \quad \bar{x}_j = 0 \quad (\text{A.15})$$

by the induction hypothesis. Using eqs. (A.12) this gives

$$x_0 = x_1 = \dots = x_k = 0. \quad (\text{A.16})$$

It follows that eqs. (4.25-4.27) provides a linearly independent condition on the coefficients  $g_{0ik-i}$ ,  $0 \leq i \leq k$ .

4. Define an  $M \times \frac{k(k+3)}{2}$  matrix  $D^{(k)}$ , whose columns are the  $M$ -component vectors

$$D_m^{(k)} = C_m^{(j,k-i-j)}; \quad 0 \leq i \leq k-1; \quad 0 \leq j \leq k-i; \quad 0 \leq m \leq M. \quad (\text{A.17})$$

The rank (number of linearly-independent columns) of  $D^{(k)}$  is  $2k$ .

*Proof:* We use induction on  $k$ . For  $k = 1$ ,  $D^{(1)}$  is a  $2 \times 2$  matrix with columns

$$D_m^{(1)} = \left( C_m^{(0,1)}, \quad C_m^{(1,0)} \right) = \frac{(-1)^{m+1}}{m+1} (1, \quad 2^m), \quad (\text{A.18})$$

that are linearly independent; the rank of  $D^{(1)}$  is 2. Let us check explicitly also the case  $k = 2$ . In this case

$$D_m^{(2)} = \left( C_m^{(0,1)}, \quad C_m^{(1,0)}, \quad C_m^{(0,2)}, \quad C_m^{(1,1)}, \quad C_m^{(2,0)} \right). \quad (\text{A.19})$$

The first two columns are the same as in the case  $k = 1$ : they span a 2-dimensional subspace. The last three columns are independent as a consequence of statement 3 of this Appendix. Furthermore,  $C_m^{(0,2)}$  and  $C_m^{(2,0)} = 2^m C_m^{(0,2)}$  are independent of all other columns, because they are the only ones that are proportional to a degree-1 polynomial in  $m$ . Finally,  $C_m^{(1,1)}$  is a linear combination of the first two columns, as a consequence of eq. (A.9) with  $i = j = 1$ . Thus, the rank of  $D^{(2)}$  is  $2 + 2 = 4$ .

We now assume that  $D^{(k-1)}$  has rank  $2(k-1)$ , and we write the columns of  $D^{(k)}$  as

$$D_m^{(k)} = \left( C_m^{(j,k-1-i-j)}, \quad C_m^{(l,k-l)} \right) \quad (\text{A.20})$$

$$0 \leq i \leq k-2, \quad 0 \leq j \leq k-1-i \quad 0 \leq l \leq k. \quad (\text{A.21})$$

By the induction hypothesis, only  $2(k-1)$  of the columns  $C_m^{(j,k-1-i-j)}$  are independent. The columns  $C_m^{(l,k-l)}$  are all independent as a consequence of statement 3; among them, those with  $1 \leq l \leq k-1$  can be expressed as linear combinations of  $C_m^{(j,k-1-i-j)}$  by eq. (A.9). Only  $C_m^{(0,k)}$  and  $C_m^{(k,0)}$  are independent of all other columns because they are proportional to a degree- $(k-1)$  polynomial in  $m$ , while all others are at most of degree  $(k-2)$ . Hence, only two independent vectors are added to the  $2(k-1)$ -dimensional subspace spanned by  $C_m^{(j,k-1-i-j)}$ , and the rank of  $D^{(k)}$  is

$$2(k-1) + 2 = 2k. \quad (\text{A.22})$$

It follows that each individual terms in the sum over  $m$  in eq. (4.7) depends only on  $2k$  independent linear combinations of the coefficients  $g_{ijk-i-j}$ ,  $0 \leq i \leq k-1$ ,  $0 \leq j \leq k-i$ .

## References

- [1] V. V. Sudakov, Sov. Phys. JETP **3** (1956) 65.
- [2] S. Catani and L. Trentadue, Nucl. Phys. B **327** (1989) 323.
- [3] G. Sterman, Nucl. Phys. B **281** (1987) 310.
- [4] H. Contopanagos, E. Laenen and G. Sterman, Nucl. Phys. B **484** (1997) 303.
- [5] J. C. Collins and D. E. Soper, Nucl. Phys. B **193** (1981) 381 [Erratum-ibid. B **213** (1983) 545].
- [6] J. C. Collins, D. E. Soper and G. Sterman, Adv. Ser. Direct. High Energy Phys. **5** (1988) 1, hep-ph/0409313.
- [7] S. Forte and G. Ridolfi, Nucl. Phys. B **650** (2003) 229 [arXiv:hep-ph/0209154].
- [8] E. Laenen, G. Oderda and G. Sterman, Phys. Lett. B **438** (1998) 173.
- [9] N. Kidonakis, G. Oderda and G. Sterman, Nucl. Phys. B **531** (1998) 365.
- [10] S. Catani, M. L. Mangano and P. Nason, JHEP **9807** (1998) 024.
- [11] R. Bonciani, S. Catani, M. L. Mangano and P. Nason, Phys. Lett. B **575** (2003) 268.
- [12] R. Bonciani, S. Catani, M. L. Mangano and P. Nason, Nucl. Phys. B **529** (1998) 424 [arXiv:hep-ph/9801375].
- [13] S. Weinberg, “*The Quantum Theory of Fields*”, Vol. I (Cambridge U. P. , Cambridge 1995) and ref. therein
- [14] P. Aurenche, A. Douiri, R. Baier, M. Fontannaz and D. Schiff, Phys. Lett. B **140** (1984) 87; P. Aurenche, R. Baier, M. Fontannaz and D. Schiff, Nucl. Phys. B **297** (1988) 661; L. E. Gordon and W. Vogelsang, Phys. Rev. D **48** (1993) 3136.